Exercise 64

(a) One way of defining $\sec^{-1} x$ is to say that $y = \sec^{-1} x \iff \sec y = x$ and $0 \le y < \pi/2$ or $\pi \le y < 3\pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$$

(b) Another way of defining $\sec^{-1} x$ that is sometimes used is to say that $y = \sec^{-1} x \iff \sec y = x$ and $0 \le y \le \pi$, $y \ne \pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

Solution

Let $y = \sec^{-1} x$. Then

$$\sec y = x. \tag{1}$$

Differentiate both sides with respect to x.

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}(x)$$

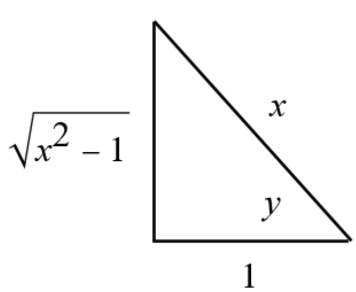
Use the chain rule to differentiate $y = y(x)$.

$$(\sec y)(\tan y)\frac{dy}{dx} = 1$$

Solve for dy/dx and write it in terms of sine and cosine.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sec y \tan y} \\ &= \frac{1}{\left(\frac{1}{\cos y}\right)\left(\frac{\sin y}{\cos y}\right)} \\ &= \frac{(\cos y)(\cos y)}{\sin y} \end{aligned}$$

Draw the implied right triangle from equation (1) and use it to determine the sine and cosine of y.



Part (a)

If $0 \leq y < \pi/2$ (Quadrant 1—sine and cosine have to be positive), then

$$\frac{dy}{dx} = \frac{(\cos y)(\cos y)}{\sin y} = \frac{\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)}{\frac{\sqrt{x^2-1}}{x}} = \frac{x}{x^2\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}.$$

If $\pi \leq y < 3\pi/2$ (Quadrant 3—sine and cosine have to be negative), then

$$\frac{dy}{dx} = \frac{(\cos y)(\cos y)}{\sin y} = \frac{\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)}{\frac{\sqrt{x^2-1}}{x}} = \frac{x}{x^2\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}.$$

Do note that since $x = \sec y$, $0 \leq y < \pi/2$ means that $1 \leq x < \infty$; and $\pi \leq y < 3\pi/2$ means that $-\infty < x \leq -1$. Therefore, with this definition,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}.$$

Part (b)

If $0 \leq y < \pi/2$ (Quadrant 1—sine and cosine have to be positive), then

$$\frac{dy}{dx} = \frac{(\cos y)(\cos y)}{\sin y} = \frac{\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)}{\frac{\sqrt{x^2-1}}{x}} = \frac{x}{x^2\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}.$$

If $\pi/2 < y \leq \pi$ (Quadrant 2—sine has to be positive, and cosine has to be negative), then

$$\frac{dy}{dx} = \frac{(\cos y)(\cos y)}{\sin y} = \frac{\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)}{\frac{\sqrt{x^2-1}}{|x|}} = \frac{|x|}{x^2\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}.$$

Do note that since $x = \sec y$, $0 \leq y < \pi/2$ means that $1 \leq x < \infty$; and $\pi/2 < y \leq \pi$ means that $-\infty < x \leq -1$. Therefore, with this definition, the two formulas for dy/dx can be combined as

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

because $x = |x|$ on $1 \leq x < \infty$.